LECTURE NO 11

CURL OF A VECTOR AND STOKES'S THEOREM

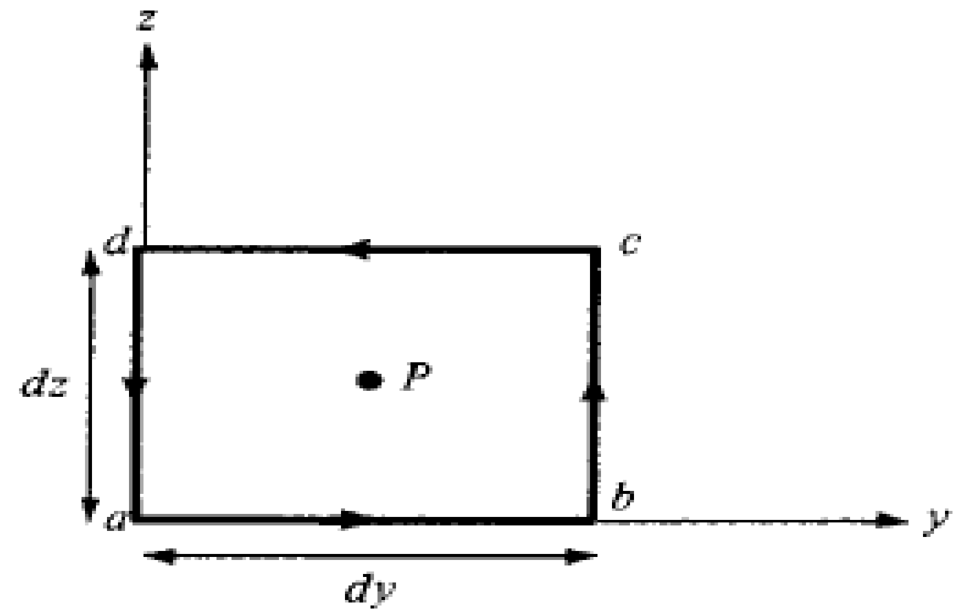
The **curl** of **A** is an axial (or rotational) vector whose magnitude is the maximum circulation of **A** per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented so as to make the circulation maximum.[2]

To obtain an expression for $\nabla \times \mathbf{A}$ from the definition in eq. (3.45), consider the differential area in the yz-plane as in Figure 3.18. The line integral in eq. (3.45) is obtained as

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \left(\int_{ab} + \int_{bc} + \int_{cd} + \int_{da} \right) \mathbf{A} \cdot d\mathbf{l} \tag{3.46}$$

We expand the field components in a Taylor series expansion about the center point $P(x_o, y_o, z_o)$ as in eq. (3.34) and evaluate eq. (3.46). On side ab, $d\mathbf{l} = dy\,\mathbf{a}_y$ and $z = z_o - dz/2$, so

$$\int_{ab} \mathbf{A} \cdot d\mathbf{l} = dy \left[A_y(x_o, y_o, z_o) - \frac{dz}{2} \frac{\partial A_y}{\partial z} \bigg|_P \right] \tag{3.47}$$


z
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On side da, $d\mathbf{l} = dz\,\mathbf{a}_z$ and $y = y_o - dy/2$, so

$$\int_{da} \mathbf{A} \cdot d\mathbf{l} = -dz \left[A_z(x_o, y_o, z_o) - \frac{dy}{2} \frac{\partial A_z}{\partial y} \bigg|_P \right] \tag{3.50}$$

Substituting eqs. (3.47) to (3.50) into eq. (3.46) and noting that $\Delta S = dy\,dz$, we have

$$\lim_{\Delta S \to 0} \oint_L \frac{\mathbf{A} \cdot d\mathbf{l}}{\Delta S} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

or

$$(\text{curl } \mathbf{A})_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \tag{3.51}$$

The y- and x-components of the curl of **A** can be found in the same way. We obtain

$$(\text{curl } \mathbf{A})_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \tag{3.52a}$$

$$(\text{curl } \mathbf{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \tag{3.52b}$$

The definition of $\nabla \times \mathbf{A}$ in eq. (3.45) is independent of the coordinate system. In Cartesian coordinates the curl of **A** is easily found using

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{3.53}$$

or